Dvostruki integrali (Drumski saobraćaj)
(dodatni materijal)

Pr 3 (iz knjige) Naći masu ploče S ograničene krivama:

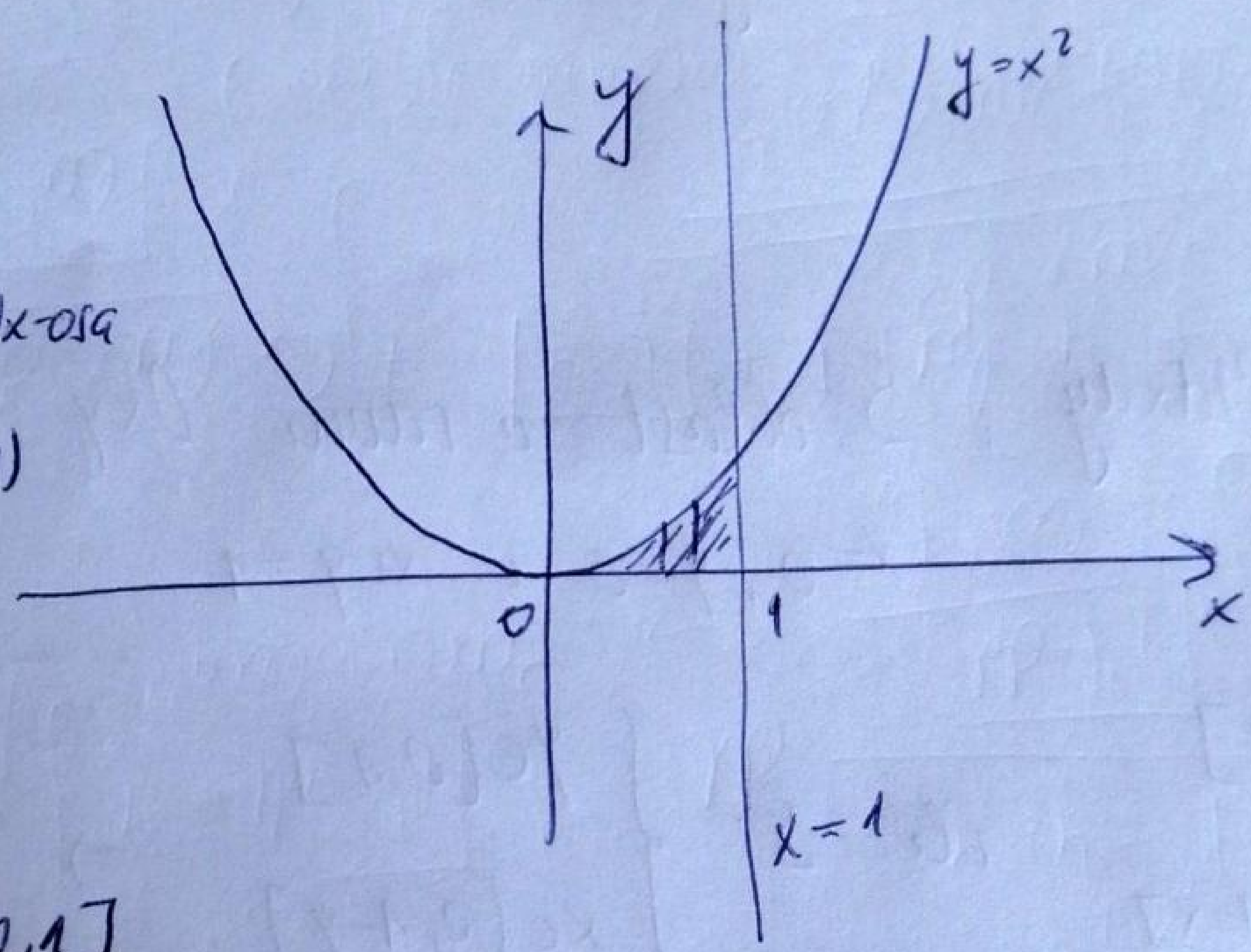
$x=1;\ y=0,\ y=x^2,\ x\geq 0$, ako je gustina materije u svakoj tački jednaka kvadratu rastojanja te tačke od koord. početka.

Rj.

$x=1$ (prava)

$y=0$ (prava) Ox-osa

$y=x^2$ (parabola)



Oblast $S: \begin{cases} x\in[0,1] \\ y\in[0,x^2] \end{cases}$

$$m=\iint_S \rho(x,y)\,dx\,dy\ ;\quad \rho(x,y) \text{ je gustina } = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$$

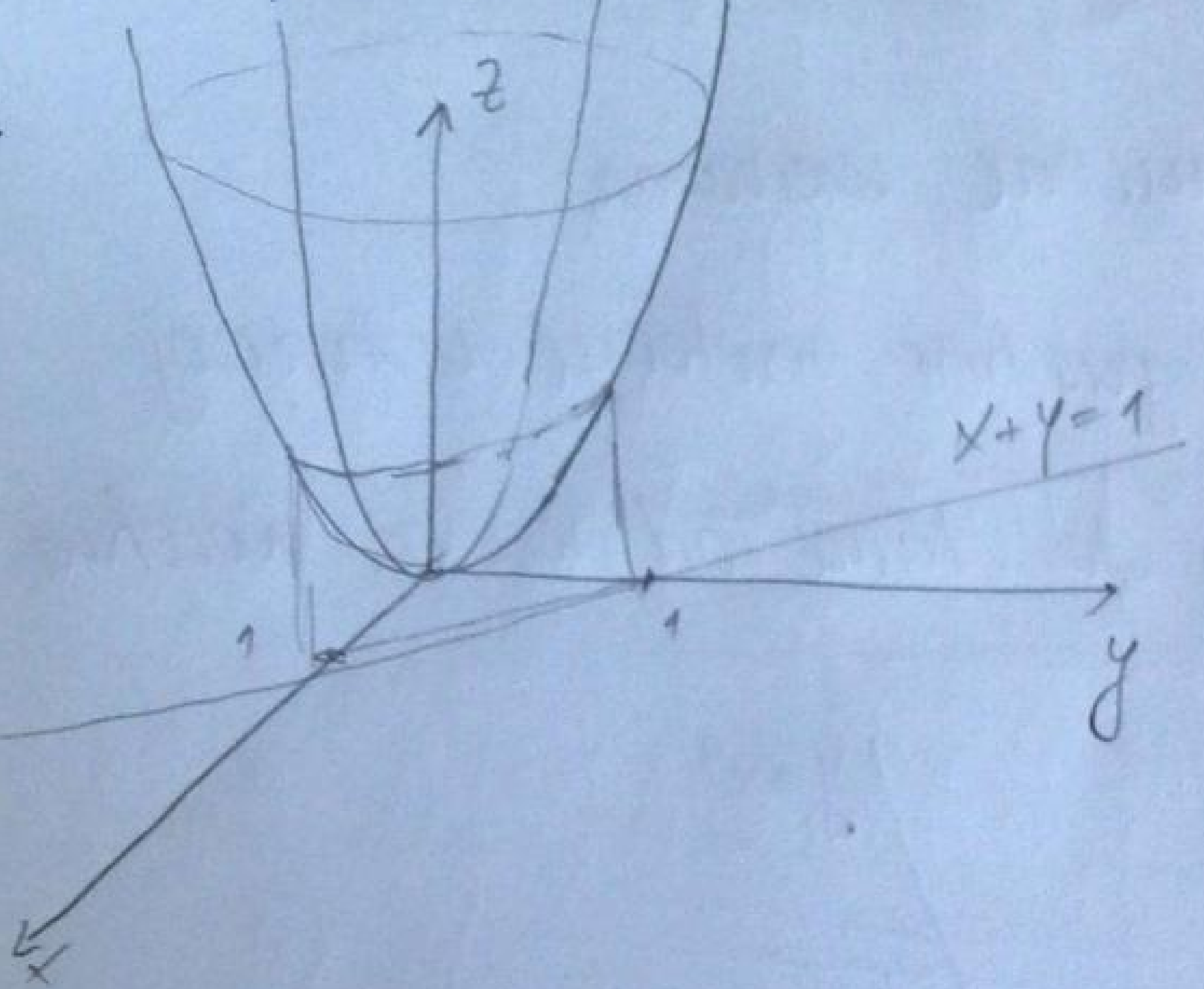
$$m=\int_0^1 dx\int_0^{x^2}(x^2+y^2)\,dy = \int_0^1 dx\left(x^2y+\frac{y^3}{3}\right)\Big|_0^{x^2} =$$

$$=\int_0^1\left(x^4+\frac{x^6}{3}-0\right)dx = \left[\frac{x^5}{5}+\frac{1}{3}\frac{x^7}{7}\right]\Big|_0^1 = \frac{1}{5}+\frac{1}{21}-0 =$$

$$=\boxed{\frac{26}{105}}$$

Pr 4) (iz knjige) Naći zapreminu tijela ograničenog paraboloidom $z = x^2+y^2$, koordinatnim ravnima i ravni $x+y=1$.

Rj



$V = \iint_S (x^2+y^2)\,dx\,dy$, S oblast u ravni Oxy ograničena sa $x=0,\ y=0$ i $x+y=1$

$$S: \begin{cases} x\in[0,1] \\ y\in[0;\ 1-x] \end{cases} \quad \text{ili} \quad S: \begin{cases} y\in[0,1] \\ x\in[0, 1-y] \end{cases} \Rightarrow$$

$$V = \int_0^1 dx \int_0^{1-x} (x^2+y^2)\,dy = \int_0^1 dx \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^{1-x} =$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} \right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]\Big|_0^1 + \frac{1}{3}\int_0^1 (1-x)^3 dx =$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} I_1 = \frac{1}{12} + \frac{1}{3}\frac{1}{4} = \frac{2}{12} = \boxed{\frac{1}{6}}$$

$$I_1 = \int_0^1 (1-x)^3 dx = \left\lceil \begin{array}{l} 1-x=t \\ -dx=dt \end{array} \right. = -\int_1^0 t^3 dt = -\frac{t^4}{4}\Big|_1^0 = \frac{1}{4}$$

x	0	1
t	1	0

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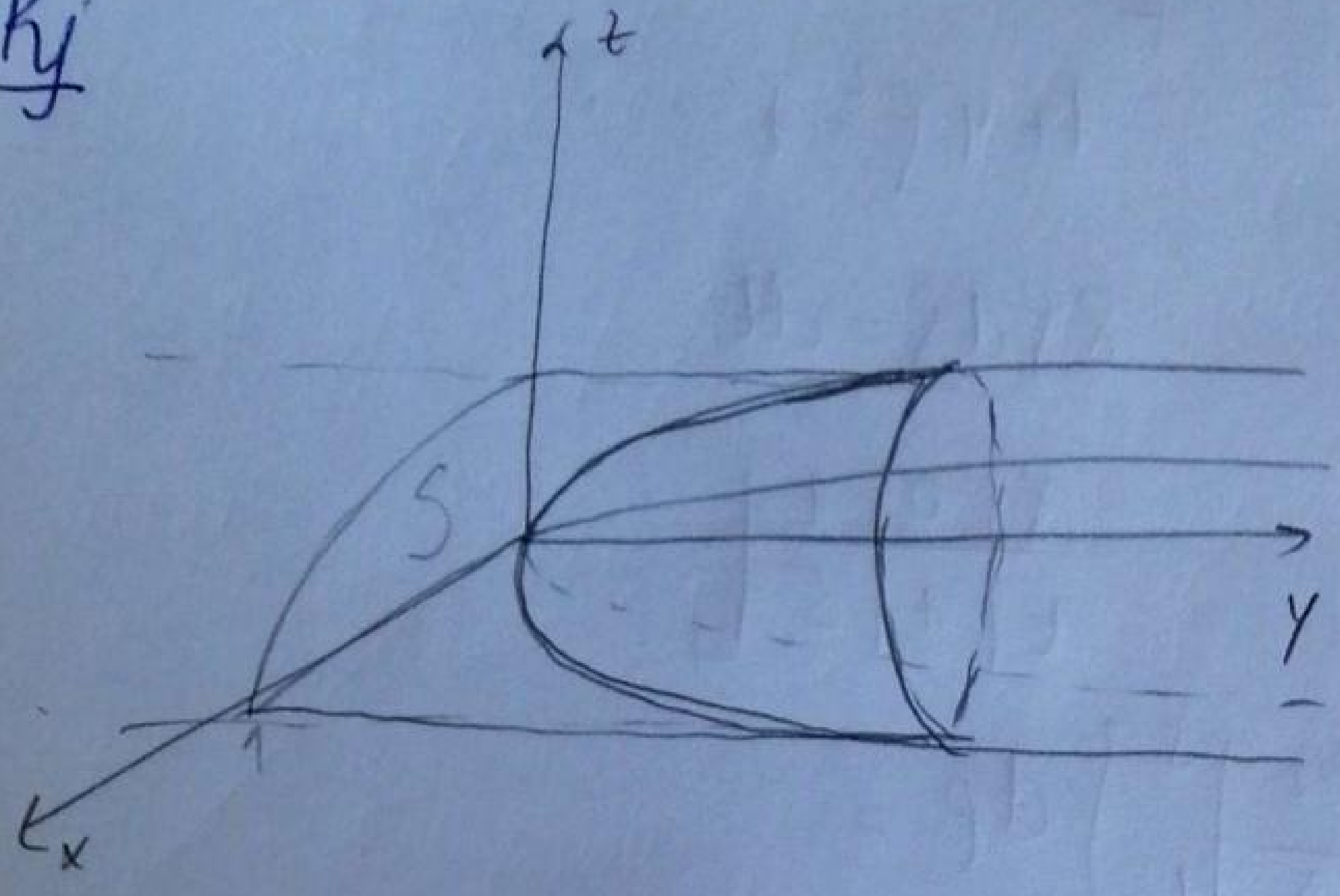
pr 11 | Izračunati površinu dijela paraboloida $y=x^2+z^2$ koji isijeca cilindar $x^2+z^2=1$ u I oktantu.

Rj:



$y=x^2+z^2=y(x,z)$

$y'_x=2x$

$y'_z=2z$

$$P = \text{formula} = \iint_S \sqrt{1+(y'_x(x,y))^2+(y'_z(x,y))^2}\,dx\,dz =$$

$$= \iint_S \sqrt{1+4x^2+4z^2}\,dx\,dz = \left\lceil S = \begin{cases} x\in[0,1] \\ z\in[0,\sqrt{1-x^2}] \end{cases} \right\rfloor =$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1+4x^2+4z^2}\,dz$$

Uvedimo polarne koordinate: $x=\rho\cos\varphi$

$z=\rho\sin\varphi$

$J=\rho$; $0\le\rho\cos\varphi\le 1$ $\quad 0\le\rho\sin\varphi\le\sqrt{1-x^2}$

$x^2+z^2=1 \Rightarrow \rho^2\cos^2\varphi+\rho^2\sin^2\varphi=1$

$\rho^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}=1$

$\underline{\rho^2=1} \Rightarrow \underline{\rho=1}$

$$S' = \begin{cases} 0\le\rho\le 1 \\ 0\le\varphi\le\frac{\pi}{2} \end{cases}$$

$$P=\int_0^{\pi/2} d\varphi \int_0^1 \rho\sqrt{1+4\rho^2\cos^2\varphi+4\rho^2\sin^2\varphi}\,d\rho =$$

$$=\int_0^{\pi/2} d\varphi \int_0^1 \rho\sqrt{1+4\rho^2}\,d\rho = \left[\begin{array}{l} 1+4\rho^2=t \\ 8\rho\,d\rho=dt \end{array}\right] =$$

ρ	0	1
t	1	5

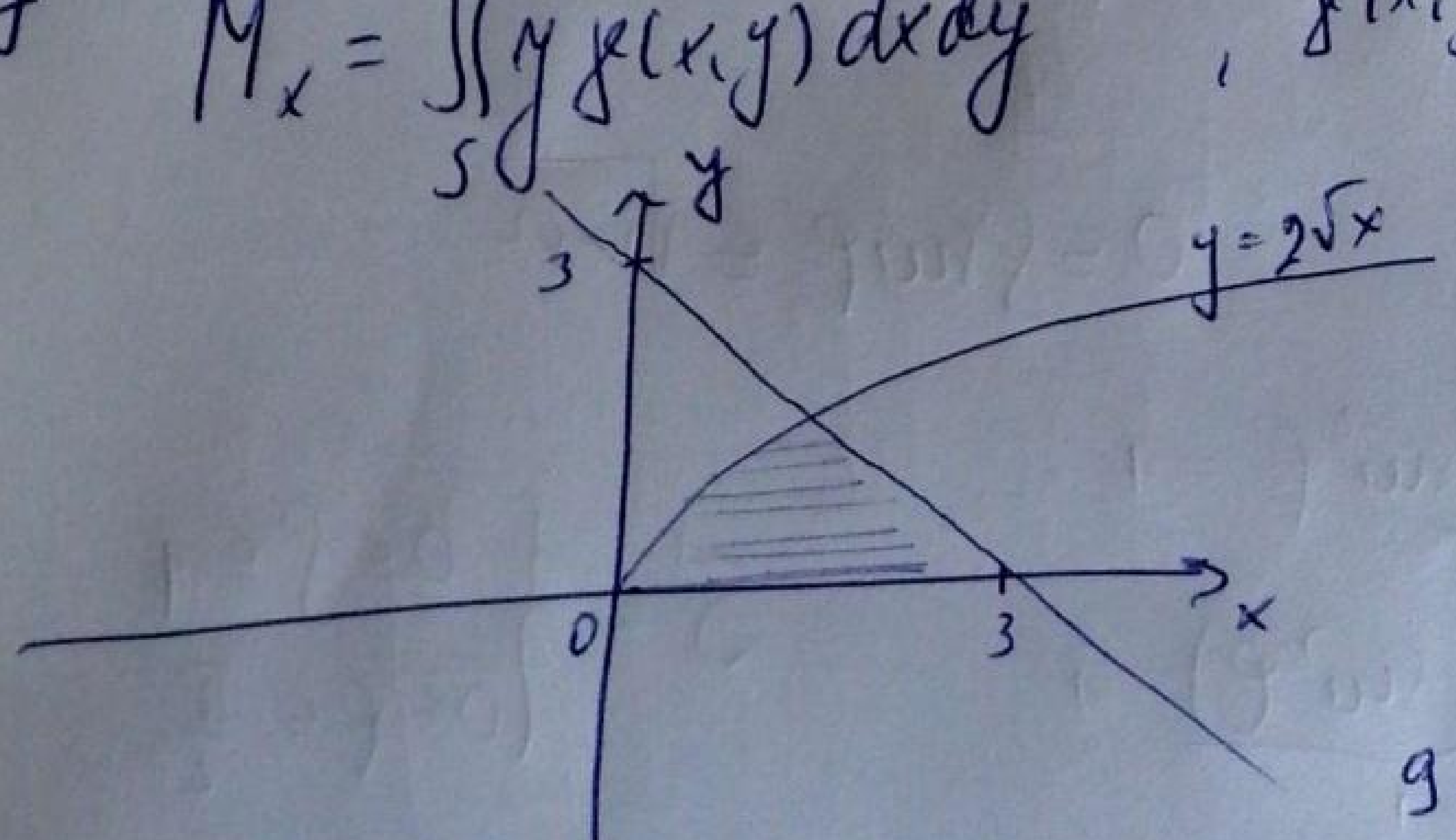
$$=\int_0^{\pi/2}\frac{1}{8}d\varphi\int_1^5\sqrt{t}\,dt=\frac{1}{8}\int_0^{\pi/2}\left[\frac{2t^{3/2}}{3}\Big|_1^5\right]d\varphi=$$

$$=\frac{1}{8}\int_0^{\pi/2}\left(\frac{2}{3}(5\sqrt5-1)\right)d\varphi=\frac{2}{24}(5\sqrt5-1)\,\varphi\Big|_0^{\pi/2}=$$

$$=\frac{1}{12}(5\sqrt5-1)\cdot\frac{\pi}{2}=\boxed{\frac{1}{24}(5\sqrt5-1)\pi}$$

nr 13 | (iz knjige) Izračunati statičke momente homogene materijalne ploče S ograničene linijama: $y=2\sqrt{x}$, $x+y=3$, $y=0$ u odnosu na osu Ox.

Rj. $M_x=\iint_S y\,\gamma(x,y)\,dx\,dy$, $\gamma(x,y)=1$ (smatramo da je gustina materije 1).



presjek $\begin{cases} y=2\sqrt{x} \\ x+y=3 \Rightarrow y=3-x \end{cases}$

$3-x=2\sqrt{x}\ \big|^2$

$9-6x+x^2=4x$

$x^2-10x+9=0 \Rightarrow x_{1,2}=\dfrac{10\pm\sqrt{\underbrace{100-36}_{=8}}}{2}$

$x_1=9\quad x_2=1$

$x = 1 \Rightarrow y = 3 - 1 = 2$

$\Rightarrow S = \begin{cases} y \in [0,2] \\ x \in \left[\frac{y^2}{4}, 3-y\right] \end{cases}$

$y = 2\sqrt{x}$

$\sqrt{x} = \frac{y}{2}$

$x = \frac{y^2}{4}$

$$M_x = \int_0^2 dy \int_{\frac{y^2}{4}}^{3-y} y\,dx =$$

$$= \int_0^2 dy \cdot y \cdot x \Big|_{\frac{y^2}{4}}^{3-y} =$$

$$= \int_0^2 \left[(3-y)y - \frac{y^3}{4}\right] dy = \left(3\frac{y^2}{2} - \frac{y^3}{3} - \frac{1}{4}\frac{y^4}{4}\right)\Big|_0^2 =$$

$$= \frac{3}{2}\cdot 4 - \frac{8}{3} - \frac{16}{16} = \frac{7}{3}.$$

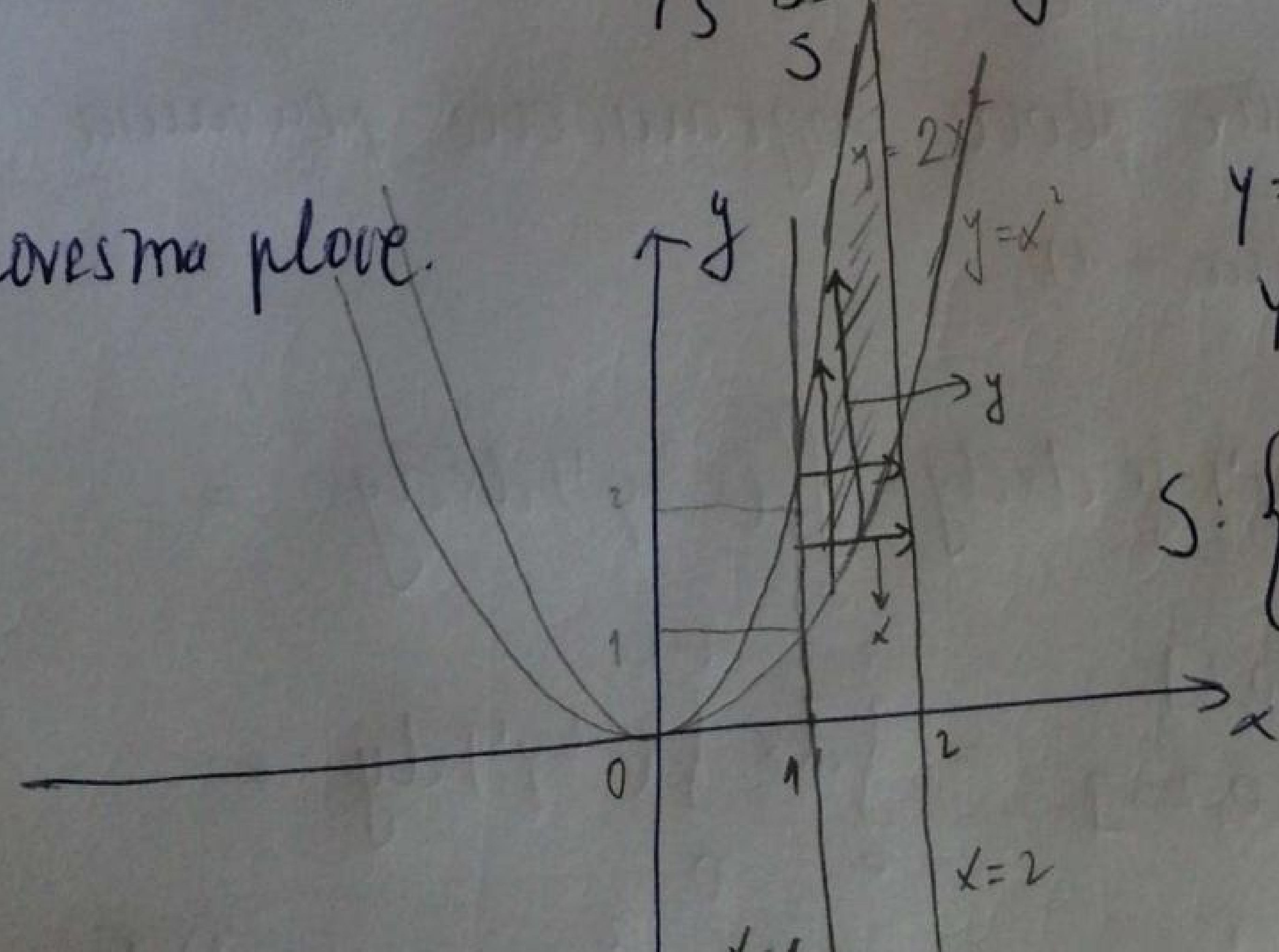
Pr15] Naći koordinate težišta homogene materijalne ploče ograničene sa: $y = x^2$, $y = 2x^2$, $x = 1$, $x = 2$.

Rj:

Pretp. da je gustina $\rho(x,y) = 1$.

$T(x_T, y_T)$; $x_T = \frac{1}{P_S}\iint_S x\,dx\,dy$, $y_T = \frac{1}{P_S}\iint_S y\,dx\,dy$.

P_S - površina ploče.



$y = x^2 \rightarrow$ parabola

$y = 2x^2 \rightarrow$ -||-

$S: \begin{cases} x \in [1,2] \\ y \in [x^2, 2x^2] \end{cases}$

$$P_S = \iint_S dx\,dy = \int_1^2 dx \int_{x^2}^{2x^2} dy = \int_1^2 dx\,(y)\Big|_{x^2}^{2x^2} =$$

$$= \int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

$$X_T = \frac{3}{7}\iint_S x\,dx\,dy = \frac{3}{7}\int_1^2 x\,dx \int_{x^2}^{2x^2} dy = \frac{3}{7}\int_1^2 x\,dx \cdot \overbrace{x^2}^{x^3} = \frac{3}{7}\,\frac{x^4}{4}\Big|_1^2 =$$

$$= \frac{3}{28}(16-1) = \frac{3\cdot 15}{28} = \frac{45}{28}$$

$$Y_T = \frac{3}{7}\iint_S y\,dx\,dy = \frac{3}{7}\int_1^2 dx \int_{x^2}^{2x^2} y\,dy = \frac{3}{7}\int_1^2 \frac{y^2}{2}\Big|_{x^2}^{2x^2} dx =$$

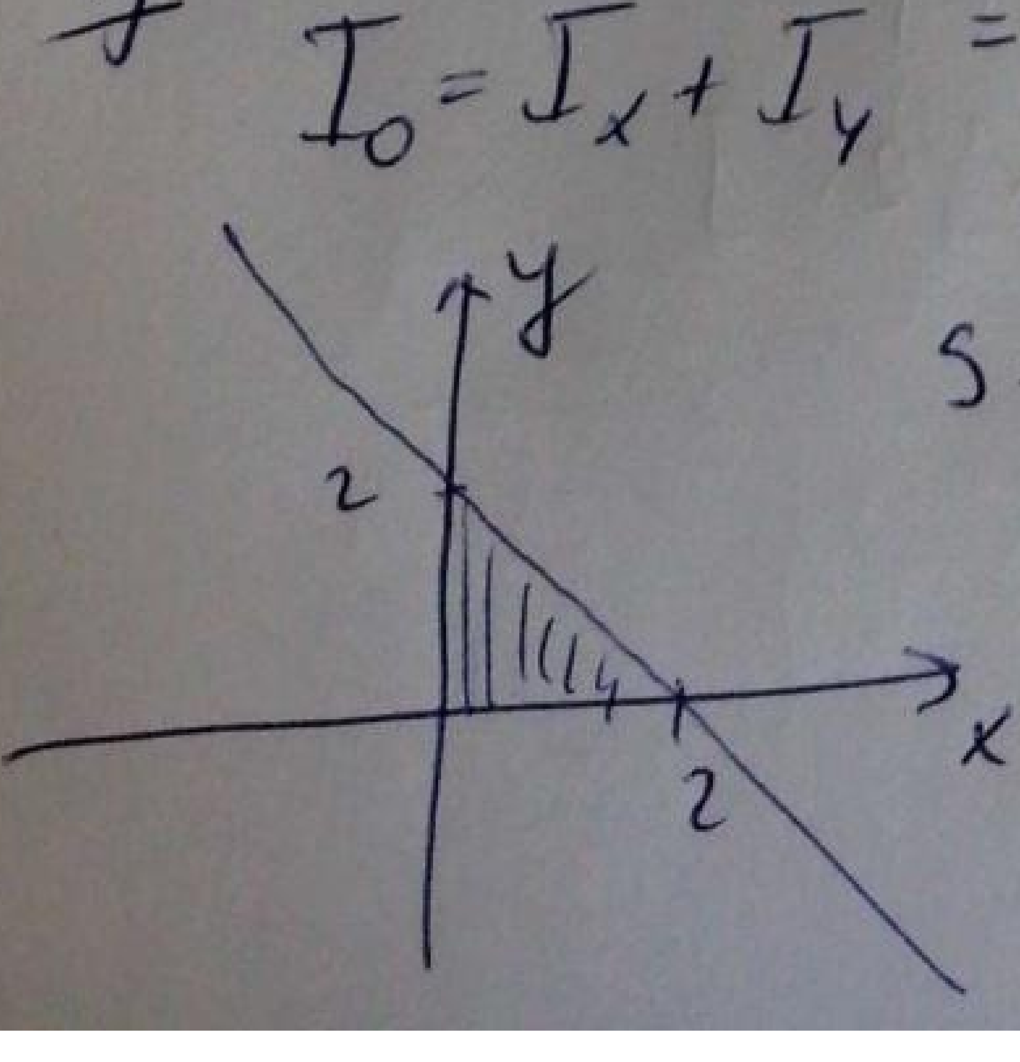
$$= \frac{3}{7}\,\frac{1}{2}\int_1^2 3x^4\,dx = \frac{9}{14}\,\frac{x^5}{5}\Big|_1^2 = \frac{9}{14\cdot 5}(32-1) = \frac{9\cdot 31}{70} = \frac{279}{70}.$$

pr 16] Izračunati moment inercije u odnosu na koord. početak ravne homogene materijalne ploče ograničene pravama:

$x+y=2,\ x=0,\ y=0$ (gustina $= 1$).

Rj

$$I_0 = I_x + I_y = \iint_S (x^2+y^2)\cdot \rho\,dx\,dy = \iint_S (x^2+y^2)\,dx\,dy =$$



$$S = \begin{cases} x \in [0,2] \\ y \in [0, 2-x] \end{cases} = \int_0^2 dx \int_0^{2-x} (x^2+y^2)\,dy =$$

$$= \int_0^2 dx\left(x^2 y + \frac{y^3}{3}\right)\Big|_0^{2-x} = \int_0^2 \left(x^2(2-x) + \frac{(2-x)^3}{3}\right)dx =$$

$$= 2\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\int_0^2 (2-x)^3\,dx = \frac{8}{3}.$$